ELSEVIER

Discrete Mathematics 226 (2001) 347–349

DISCRETE MATHEMATICS

Communication

Proof of a conjecture by Walter Deuber concerning the distances between points of two types in R^d

Dietrich Morgenstern *

Institute of Mathematical Stochastics, University of Hannover, Welfengarten1, 30167 Hannover, Germany

Received 13 July 2000; accepted 26 July 2000
Communicated by H. Sachs

Abstract

Theorem. (A) *For equal numbers of black and white points in euclidean space the sum of the pairwise distances between points of equal color is less than or equal to the sum of the pairwise distances between points of different color.*

(B) *Equality holds only in the case when black and white points coincide.*

Statement (B) was originally formulated as a question in 1998 by the late Walter Deuber (Bielefeld); his question implicitly includes the assumption (conjecture) that (A) is true.

Keywords: Finite point sets; Euclidean distances; Geometric inequalities

In 1998, Walter Deuber [1] expressed in the form of a question the following conjecture:

> For equal numbers of black and white points in euclidean space the sum of the pairwise distances between points of equal color is less than or equal to the sum of the pairwise distances between points of different color, and equality holds only in the case when black and white points coincide.

The condition for equality is the question of Deuber; the inequality is a consequence, a fact assumed to be known to Deuber.

The conjecture will be proved by simple counting and application of a well-known fact from integral geometry.

* Correspondence address: Im Dorffeld 58, D-30966 Hemmingen, Germany.

0012-365X/01/$ - see front matter
PII: S0012-365X(00)00209-0

Proof. *Part I*: Let m be the common number of black points B_i and of white points W_i. For any hyperplane not meeting any black or white points ($\sum_{i=1}^{d} a_i \xi_i = \text{const.}$), let k and l denote the numbers of black and white points, respectively, on one side of the hyperplane, leaving $m-k$ and $m-l$ black and white points, respectively, on the other side. The numbers of pairs of points with the two points on different sides of the hyperplane (each pair giving rise to one intersection point of the hyperplane with the straight-line segment connecting the two points) are

$$\begin{aligned} b &= number\ of\ black\text{–}black\ connections \\ &= k(m-k), \end{aligned}$$

$$\begin{aligned} w &= number\ of\ white\text{–}white\ connections \\ &= l(m-l), \end{aligned}$$

$$\begin{aligned} g &= number\ of\ black\text{–}white\ connections \\ &= k(m-l) + l(m-k) \end{aligned}$$

yielding

$$(*)\ \ g - (w+b) = (k-l)^2 \geqslant 0.$$

Part II: The invariant measure of hyperplanes (see the remark at the end of this paper) has the property that the measure of those intersecting a straight-line segment s is proportional to the length of s; restricting the measure to those hyperplanes meeting a ball that contains all black and white points and normalizing it results in a probability distribution which gives a probability proportional to the length. Taking the expectation of the inequality $(*)$ yields the desired inequality

$$\sum_{i,k} \mathrm{d}(B_i, W_k) - \sum_{i<k} \mathrm{d}(B_i, B_k) - \sum_{i<k} \mathrm{d}(W_i, W_k) \geqslant 0,$$

where $\mathrm{d}(A,B)$ denotes the distance between points A and B.

Part III: The case of equality requires that $k=l$ for almost all hyperplanes. Moving the hyperplane across one black point that does not coincide with a white point changes the number of black points on one side of the hyperplane from k to $k' = k \pm 1$ while l remains unchanged, $l' = l$: as $l' = l = k \neq k'$, we have a contradiction. □

1. Generalization to spheres S^d in R^{d+1}

The main inequality and the proof remain the same as long as no two points are ‘diametrical’ (in order to have uniquely defined connections between any two of the points so that, for the case of equality, the above arguments apply); instead of hyperplanes, the intersections of hyperplanes through the center of the sphere with the sphere are used.

2. Addenda

(1) Ulrich Brehm (Technical University of Dresden) had the idea of proving Deuber's conjecture in a similar way; he intends to publish all his results (going beyond this special one) in the near future.

(2) As Horst Sachs (Technical University of Ilmenau) found first, the main inequality holds also in the case of $m+1$ black points and m white points; in the first part of the proof above the left-hand side of the essential inequality $(*)$ turns into

$$g-(w+b)=(k-l)^2-(k-l),$$

which is still non-negative since $k-l$ is an integer.

He also showed that equality holds if and only if the points that remain after all pairs of differently colored coinciding points have been omitted from a black–white alternating sequence on a straight line (note that the case $m=1$ corresponds to the triangle inequality). Also, the use of weights w_i instead of two colors for all points (with $\sum_i w_i=0$) leads to generalizations which will be contained in a forthcoming paper by H. Sachs.

3. The invariant measure for hyperplanes

The invariant probability distribution for hyperplanes hitting a fixed ball, as used in the proof above, can be obtained this way: connect the center of the ball with a random point on the sphere which is the 'surface' of the ball, uniformly distributed, and get a random point on this radius, also uniformly distributed on the radius, and obtain the hyperplane orthogonal to the radius through that random point.

References

[1] W. Deuber, 2. Problem of W. Deuber. Problem 303, Discrete Math. 192 (1998) 348.